Traffic flow models with 'slow-to-start' rules

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Abstract

We investigate two models for traffic flow with modified acceleration ('slow-to-start') rules. Even in the simplest case $v_{max} = 1$ these rules break the 'particle-hole' symmetry of the model. We determine the fundamental diagram (flow-density relationship) using the so-called car-oriented mean-field approach (COMF) which yields the exact solution of the basic model with $v_{max} = 1$. Here we find that this is no longer true for the models with modified acceleration rules, but the results are still in good agreement with simulations. We also compare the effects of the two different slow-to-start rules and discuss their relevance for real traffic. In addition, in one of these models we find a new phase transition to a completely jammed state.

Keywords: Cellular automata, traffic flow, improved mean-field theories

1 Introduction

The investigation of traffic flow has attracted a lot of interest from physicists in the last few years. In addition to more traditional methods borrowed e.g. from hydrodynamics, ideas from statistical physics have become more and more important. A comprehensive review of the different approaches can be found in [1].

Here we use a cellular automaton (CA) [2] which has been shown to reproduce at least qualitatively the features of real traffic [3, 4]. The basic idea is to formulate a model in discrete space and time. The road is divided into cells which can either be empty or occupied by exactly one car. The cars are characterized by an internal parameter $v = 0, 1, \ldots, v_{max}$ which corresponds to their momentary velocity. Starting from a given configuration at a time t the configuration at the next time step $t+1$ can be obtained by application of four rather simple rules which are applied to all cars in parallel: acceleration, slowing down due to other cars, randomization and the actual car motion.

The stochastic nature of the model which enters through the randomization process seems to be important for a realistic description of traffic. It is quite surprising that such a simple rule can mimic several effects which seem to be important for the behaviour of real traffic, e.g. overreactions at braking or fluctuations in driving.

Due to their inherent discrete nature CA are ideal for large-scale computer simulations. On the other hand, these models can not be described easily using analytic approaches. In [5] we solved the special case of $v_{max} = 1$ exactly. We calculated the so-called fundamental diagram, i.e. the relation between flow (current) and the density c of cars. The case $v_{max} = 1$ is special since the model is particle-hole symmetric and therefore the fundamental diagram is also symmetric with respect to $c = 1/2$ where also the flow takes its maximum value. From measurements on real traffic one knows, however, that the maximum flow is shifted to much lower densities. This fact is reproduced by the models with $v_{max} > 1$, but unfortunately these models are no longer exactly solvable.

In [7, 8] two models with slightly modified acceleration rules have been introduced. One effect of these new rules is a breaking of the particle-hole symmetry in the case $v_{max} = 1$. In this paper we want to address the question whether these modified versions are still exactly solvable or if the exact solvability is somehow connected to the existence of the particle-hole symmetry. Another interesting point is a comparision of the two rules. Both are designed to mimic the same effects observed in real traffic, but use different ways to implement these in the rules of the CA. We therefore want to compare the effects of the two different slow-to-start rules and discuss their relevance for modelling real traffic.

2 The modified models

In the following we will always assume $v_{max} = 1$. The fundamental diagram of the basic model is symmetric (maximum flow at density $c_{max} = 1/2$) due to a particle-hole symmetry [5]. In order to get a more realistic asymmetric fundamental diagram even for $v_{max} = 1$ modified models with new acceleration rules have been introduced [7, 8].

In the Benjamin-Johnson-Hui (BJH) model [7] cars which had to brake due to the next car ahead will move on the next opportunity only with probability $1 - p_s$. Each cell can be in one of four states. Either it is empty (state 'e') or it is occupied by a car with velocity $v = 1$ (state '1'), a car with velocity $v = 0$ which is static due to the action of the randomization step or the slow-to-start rule (state 's'), or a car with velocity $v = 0$ which is standing due to blocking by a car just in front of it (state 'b').

1) **Acceleration:** All cars are assigned a velocity of '1'.
2) **Slow-to-start rule:** All cars that are legitimate candidates decelerate to 's' with probability p_s.
3) **Blockage:** All cars which have no free cell in front and are thus blocked from moving change to state 'b'.
4) **Randomization:** All cars still in state '1' decelerate to state 's' with probability p.
5) **Car motion:** All cars in state '1' move one cell ahead.

In the Takayasu (T^2) model [8] a standing car will move only if there are at least two free cells in front of it. More generally it will move only with probability $1 - p_t$ if there is just one free cell ahead. Here every cell is one of three states, i.e. empty, or occupied by a car with velocity $v = 0$ or $v = 1$.

1) **Acceleration:** Standing cars accelerate to velocity $v = 1$ if there are at least two empty cells in front.
2) **Slow-to-start rule:** Standing cars with just one free cell in front accelerate only with probability $q_t = 1 - p_t$ to velocity $v = 1$.
3) **Slowing down (due to other cars):** If the distance d to the next car ahead is not larger than v ($d \leq v$) the speed is reduced to $d - 1$ [$v \stackrel{d \leq v}{\rightarrow} d - 1$].
4) **Randomization:** With probability p, the velocity of a vehicle (if greater than zero) is decreased by one [$v \stackrel{p}{\rightarrow} v - 1$].
5) **Car motion:** Each vehicle is advanced v cells.

The idea behind the modified rules of the BJH and T^2 models is to mimic the delay of a car in restarting, i.e. due to a slow pick-up of engine or loss of the driver's attention. Both modifications break the particle-hole symmetry and should yield fundamental diagrams with a maximum at a density $c_{max} < 1/2$.

In order to investigate the effects of the modified rules analytically we apply the so-called "car-oriented mean-field theory" (COMF) [6]. This approach turned out to yield the exact solution for the standard rules ($p_s = 0$, $p_t = 0$) [6]. It seems therefore worthwhile to investigate whether one also obtains the exact solution for the modified models without particle-hole symmetry.

3 COMF for BJH model

We first investigate the effects of the BJH version [7] of the slow-to-start rule.

We denote the probability to find at time t (exactly) n empty cells in front of a vehicle by $P_n(t)$. The density of cars which which obey the slow-to-start rule is denoted by $\tilde{P}_1$. As in [5, 4] we change the order of the update steps to 2-3-4-1. This change has to be taken into account when calculating the flow $f(c, p, p_s)$. It has the advantage that after step 1 there are no cars with velocity 0, i.e. all cars have velocity 1.

The time evolution of the probabilities P_n can conveniently be expressed through the probability $g(t)$ ($\bar{g}(t) = 1 - g(t)$) that a car moves (does not move) in the next timestep. These probabilities are given by:

$$g(t) = q \sum_{n\geq 1} P_n(t) + q_s q \tilde{P}_1(t) = q[1 - P_0(t)] - p_s q \tilde{P}_1(t), \tag{1}$$

where we have used the normalization

$$\sum_{n\geq 0} P_n(t) + \tilde{P}_1(t) = 1. \tag{2}$$

The probabilities can be related to the density $c = N/L$ of cars. Since each car which has the distance n to the next one in front of him 'occupies' $n+1$ cells we have the following relation:

$$\sum_{n\geq 0} (n+1) P_n(t) + 2\tilde{P}_1(t) = \frac{1}{c}\ . \tag{3}$$

As described in [6] we then obtain the time evolution of the probabilities as ($q = 1-p$, $q_s = 1 - p_s$)

$$
\begin{aligned}
P_0(t+1) &= \bar{g}(t)[P_0(t) + qP_1(t) + q_s q\tilde{P}_1(t)] && (4)\\
\tilde{P}_1(t+1) &= g(t)P_0(t) && (5)\\
P_1(t+1) &= \bar{g}(t)[pP_1(t) + (p_s + q_s p)\tilde{P}_1(t) + qP_2(t)] + qg(t)[P_1(t) + q_s\tilde{P}_1(t)] && (6)\\
P_2(t+1) &= \bar{g}(t)[pP_2(t) + qP_3(t)] + g(t)[pP_1(t) + (p_s + q_s p)\tilde{P}_1(t) + qP_2(t)] && (7)\\
P_n(t+1) &= \bar{g}(t)[pP_n(t) + qP_{n+1}(t)] + g(t)[pP_{n-1}(t) + qP_n(t)] \quad (n \geq 3) && (8)
\end{aligned}
$$

Here we are mainly interested in the stationary state ($t \to \infty$) with $\lim_{t\to\infty} P_n(t) = P_n$. In [6] we used generating functions to solve a similar set of equations ($p_S = 0$). Here we adopt a more direct method since it is easy to see that (8) is solved by the Ansatz

$$\begin{aligned} P_n &= \mathcal{N} z^n, \qquad\qquad (n \geq 2) \\ z &= \frac{pg}{q\bar{g}}, \end{aligned} \tag{9}$$

where $\mathcal{N}$ is a normalization.

Using (4-7) we can express all probabilities through P_0:

$$\tilde{P}_1 = \frac{qP_0(1-P_0)}{1+p_s q P_0}, \tag{10}$$

$$g = \frac{\tilde{P}_1}{P_0} = \frac{q(1-P_0)}{1+p_s q P_0}, \tag{11}$$

$$P_1 = \frac{zP_0}{p} - q_s\tilde{P}_1, \tag{12}$$

$$\mathcal{N} = (1 - P_0 - P_1 - \tilde{P}_1)\frac{1-z}{z^2}. \tag{13}$$

The only remaining free parameter P_0 can now be related to the density c by using (3). $P_0(c)$ is given as the root in the interval $(0,1)$ of the cubic equation

$$cp_s^2q^2P_0^3 + q[qp_s^2(1-2c) + p_s(1+c) + c]P_0^2 + [qp_s(1-3c) - 2qc + 1]P_0 - pc = 0. \tag{14}$$

The flow is given by (with $q = 1 - p$)

$$f(c, p, p_s) = cg = c \cdot \frac{q(1-P_0)}{1+p_s q P_0}. \tag{15}$$

Results for the fundamental diagram and a comparison with computer simulations are shown in Figs. 1 and 2. Obviously, the COMF is no longer exact for $p_s > 0$, but it is still an excellent approximation, especially for small p_s. For fixed p (see Fig. 1) the flow $f(c, p, p_s)$ is only reduced slightly for $p_s > 0$. In addition, the density c_{max} of maximum flow is shifted towards smaller densities with increasing p_s, i.e. the fundamental diagram is no longer symmetric. This reflects the absence of the particle-hole symmetry. There is no other qualitative change of the fundamental diagram, in contrast to the T^2 model (see next section).

4 COMF for the T^2 model

The probability $g(t)$ ($\bar{g}(t) = 1 - g(t)$) that a car moves (does not move) in the next timestep is now given by:

$$g(t) = q\sum_{n\geq 1} P_n(t) + q_t q\bar{P}_1(t) = q[1 - P_0(t)] - p_t q\bar{P}_1(t), \tag{16}$$

We obtain the time evolution of the probabilities as ($q = 1 - p$, $q_t = 1 - p_t$)

$$\begin{aligned}
P_0(t+1) &= \bar{g}(t)[P_0(t) + qP_1(t) + q_t q\bar{P}_1(t)] && (17)\\
\bar{P}_1(t+1) &= \bar{g}(t)(p_t + q_t p)\bar{P}_1(t) + g(t)P_0(t) && (18)\\
P_1(t+1) &= \bar{g}(t)[pP_1(t) + qP_2(t)] + qg(t)[P_1(t) + q_t\bar{P}_1(t)] && (19)\\
P_2(t+1) &= \bar{g}(t)[pP_2(t) + qP_3(t)] + g(t)[pP_1(t) + (p_t + q_t p)\bar{P}_1(t) + qP_2(t)] && (20)\\
P_n(t+1) &= \bar{g}(t)[pP_n(t) + qP_{n+1}(t)] + g(t)[pP_{n-1}(t) + qP_n(t)] \quad (n \geq 3) && (21)
\end{aligned}$$

Here we are mainly interested in the stationary state ($t \to \infty$) with $\lim_{t\to\infty} P_n(t) = P_n$. In this case the equations (21) – which are identical with the equations (8) since the modified rules do not affect distances larger than 1 – are again solved by the Ansatz

$$\begin{aligned}
P_n &= \mathcal{N} z^n, \qquad (n \geq 2)\\
z &= \frac{pg}{q\bar{g}}, && (22)
\end{aligned}$$

where $\mathcal{N}$ is a normalization. Expressing all quantities through P_0 and $\bar{P}_1$ we find

$$\begin{aligned}
P_1 &= \frac{g}{q\bar{g}}\bar{P}_1 = \frac{z}{p}\bar{P}_1, && (23)\\
g &= q(1 - P_0) - p_t q\bar{P}_1, && (24)\\
\mathcal{N} &= (1 - P_0 - P_1 - \bar{P}_1)\frac{1-z}{z^2}. && (25)
\end{aligned}$$

P_0 and $\bar{P}_1$ can be related through

$$p_t x\bar{P}_1^2 - (1 + pq_t(1 - P_0))\bar{P}_1 + (1 - P_0)P_0 = 0 \qquad (26)$$

which yields explicitly

$$\bar{P}_1 = \frac{1}{2p_t x}\left[1 + pq_t(1 - P_0) - \sqrt{(1 + pq_t(1 - P_0))^2 - 4p_t x(1 - P_0)P_0}\right] \qquad (27)$$

where we have used the abbreviation $x = p_t + q_t p = 1 - qq_t$.

The relation with the density is

$$y(c - 1) + (p + (1 + q)y)c(1 - P_0) - (1 + p - py)c\bar{P}_1 = 0 \qquad (28)$$

with $y = P_0 + p_t\bar{P}_1$. After solving (28) for $P_0(c)$, $\bar{P}_1(c)$ the flow can be obtained using $f(c, p, p_t) = cg$. In Figs. 3 and 4 we show the results of COMF and computer simulations. Again the COMF is no longer exact for $p_t > 0$, but it is still a good approximation. For $p_t \ll 1$ the situation is somewhat similar to the BJH model, i.e. the flow is reduced slightly and the c_{max} is shifted towards smaller densities. For large p_t, however, one finds a qualitative change of the fundamental diagram. For $p_t \lesssim 1$

the fundamental diagram has an inflection point at a density $c > c_{max}$, in contrast to the case $p_t \ll 1$, where the flow vs. density relation is convex. The parameter p_t therefore controls the curvature of the fundamental diagram.

This can be seen most clearly for $p_t = 1$. Here the flow $f(c, p, p_t)$ vanishes for $c \geq 1/2$. For $c = 1/2$ the stationary state consists of standing cars with exactly one empty site in between them ($0.0.0.0.\cdots$ where '0' denotes a standing car and '.' an empty site). All cars cannot move due to the slow-to-start rule. For larger densities the stationary state is essentially of the same form, but larger clusters of standing cars will appear. In Monte Carlo simulations starting from a mega-jam the relaxation into this stationary state is extremly slow, even for 'large' values of p, e.g. $p = 0.5$. Note that the COMF solution (22)-(28) predicts a phase transition to the completely jammed state at $c = 2/3$. There is, however, a second solution† of the COMF equations with $P_0 = 1 - \bar{P}_1 = \frac{2-c}{c}$ and $P_n = 0$ for $n \geq 1$, corresponding to $f(c, p, p_t = 1) = 0$ for $c \geq 1/2$.

The new phase transition for $p_t = 1$ only exists for $p > 0$. In [8, 9] mainly the deterministic case $p = 0$ has been investigated. There the equilibrium state depends on the initial conditions and therefore the completely jammed phase for $c \geq 1/2$ could not been found. More detailed results for the model with $p_t = 1$ will be presented in a future publication.

5 Discussion

In this paper we have investigated two cellular automata for traffic flow with modified acceleration rules. For $v_{max} = 1$ these rules break the particle-hole symmetry of the model. We used the so-called Car-Oriented Mean-Field theory to obtain an analytic desription of these models. In contrast to the case of the basic model [6] the COMF does no longer produce an exact solution of the modified versions.

The main effects of the slow-to-start rules are a (small) reduction of the flow and a shift of the location c_{max} of the maximum flow towards smaller densities. Note, however, that the reduction in the T^2 model is larger than in the BJH model, especially for large p_t. For $p_t = 1$ we found a phase transition to a completely jammed phase with flow $f(c, p, p_t = 1) = 0$ near $c = 1/2$. This transition will be investigated in more detail in a future publication.

The existence of this transition reflects the most important difference between the two slow-to-start rules. The BJH rule is of *temporal* nature (the restart probability depends on the number of attempts) whereas the T^2 rule is of *spatial* nature (the restart probability depends on the number of empty cells in front of the car).

In Figs. 5 and 6 we show the numbers $\tilde{P}_1(c)$ and $\bar{P}_1(c)$ of cars affected by the slow-to-start rule in the BJH and T^2 model, respectively. As expected the modified rules

† This solution corresponds to (27) with a '+' in front of the square-root.

are most effective for densities slightly larger than the density of maximum flow in the original model, i.e. for $c > 1/2$. This effect is most pronounced for small values of p.

Comparing the effects of the two modifications, the T^2 model appears to be more interesting since the effects are larger. An interesting feature of this model is that the curvature of the fundamental diagram for densities $c > 1/2$ may vary, depending on the value of p_t. The idealized shape of the fundamental diagram has been discussed controversely in the literature (see e.g. [10]) and there exists experimental evidence for fundamental diagrams with and without inflection points. The slow-to-start rule of T^2 offers a simple explanation for these observations.

In the present paper we only investigated the case $v_{max} = 1$. The main modification introduced by the slow-to-start rules is a asymmetry between acceleration and deceleration processes. Since this is already included in the basic model for $v_{max} > 1$ we do not expect that the modifications might have a drastic effect for larger velocities. This expectation seems to be justified as preliminary results from computer simulations show. For the T^2 model, however, the slow-to-start probability p_t still controls the curvature of the fundamental diagram.

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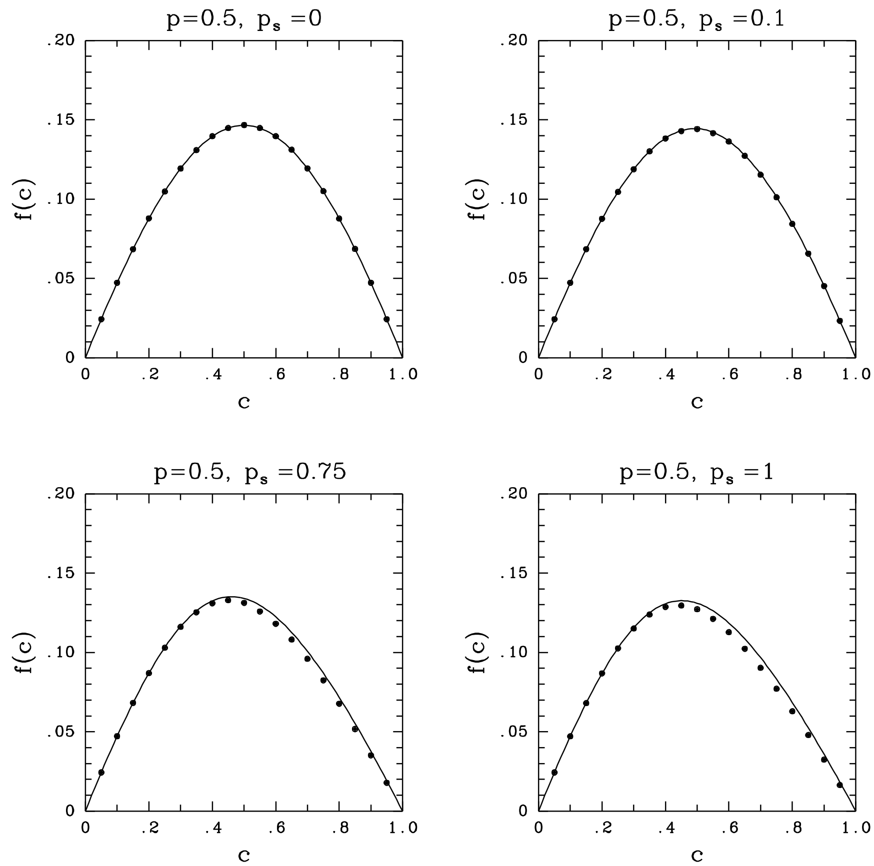

Figure 1: Fundamental diagram for the BJH model with $p = 0.5$ and different p_s. The full line is the COMF result. For comparison the results from computer simulations ($\bullet$) are also shown.

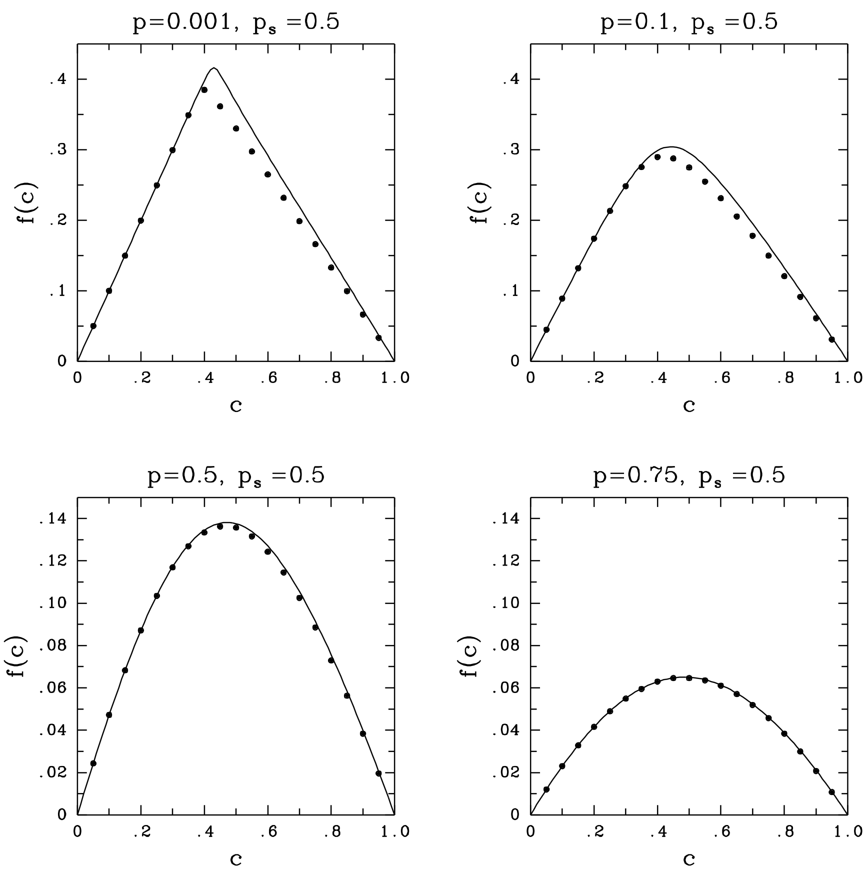

Figure 2: Same as Fig. 1, but for $p_s = 0.5$ and different p.

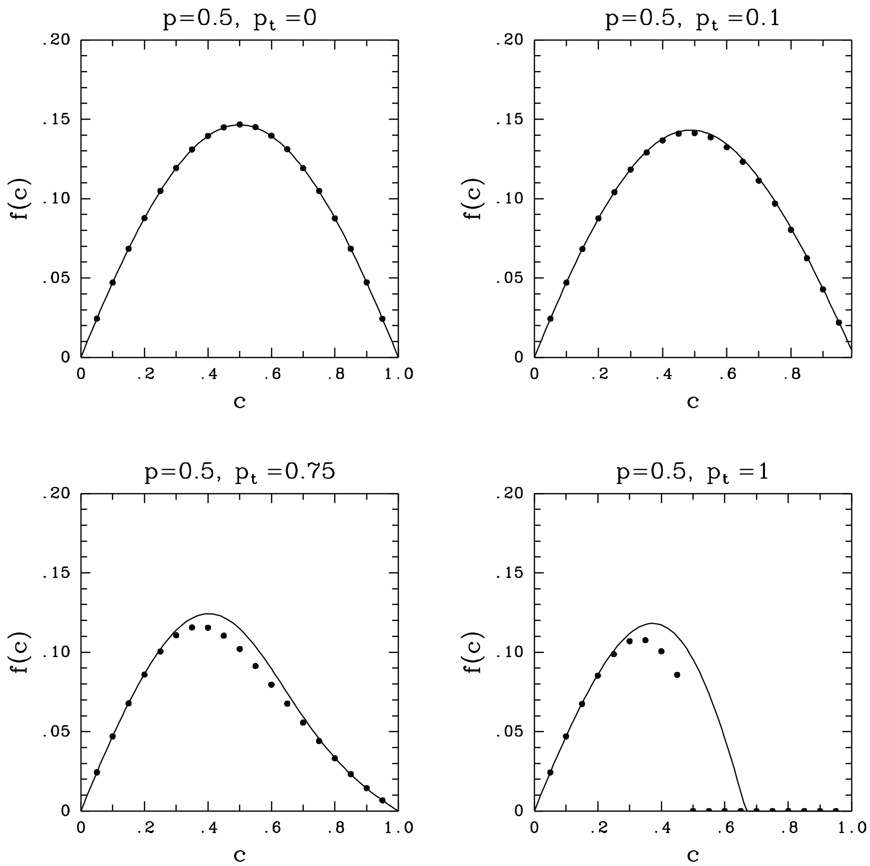

Figure 3: Fundamental diagram for the T^2 model with $p = 0.5$ and different p_s. The full line is the COMF result. For comparison the results from computer simulations ($\bullet$) are also shown.

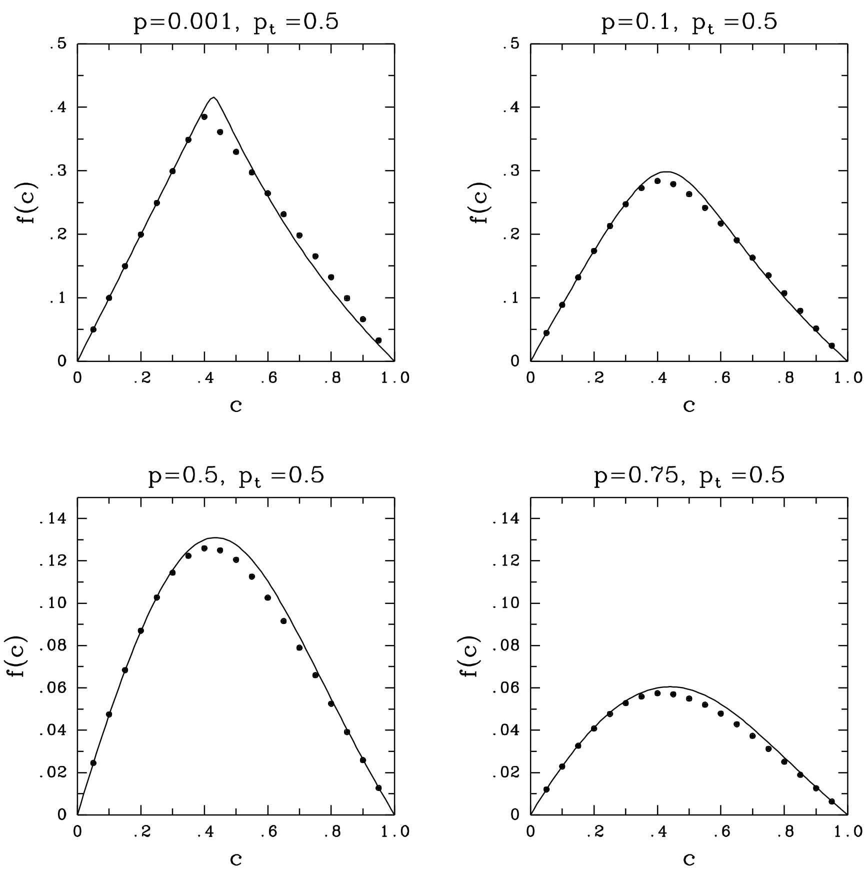

Figure 4: Same as Fig. 3, but for $p_s = 0.5$ and different p.

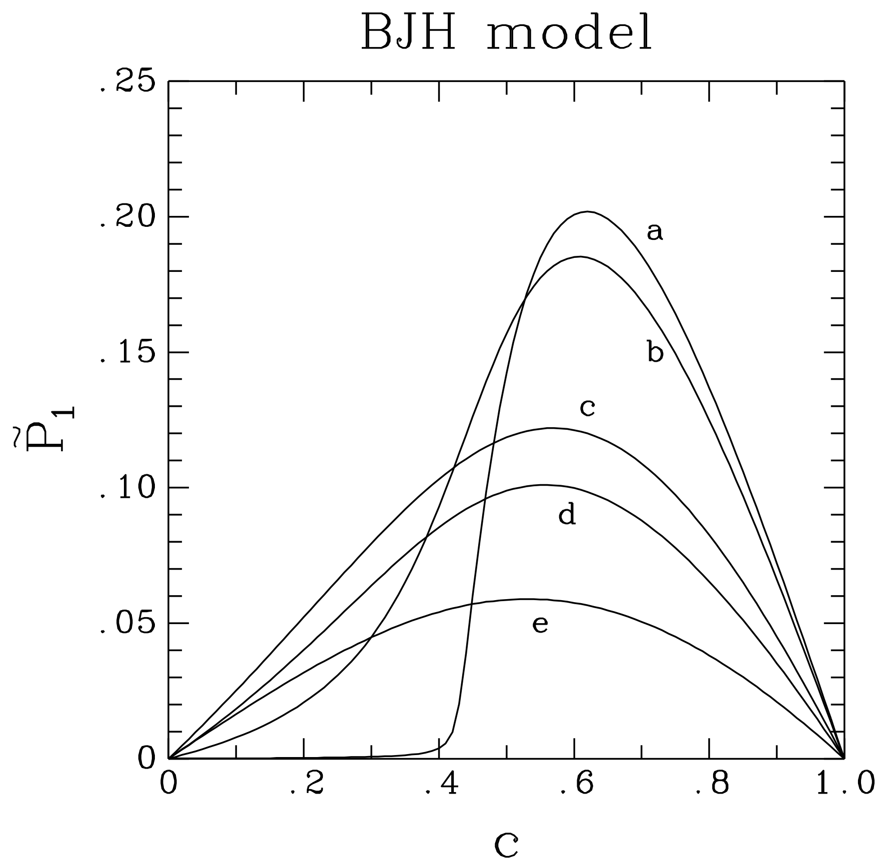

Figure 5: $\tilde{P}_1$ for the BJH model for a) $p = 0.001$, $p_s = 0.5$, b) $p = 0.1$, $p_s = 0.5$, c) $p = 0.5$, $p_s = 0.1$, d) $p = 0.5$, $p_s = 1$, e) $p = 0.75$, $p_s = 0.5$.

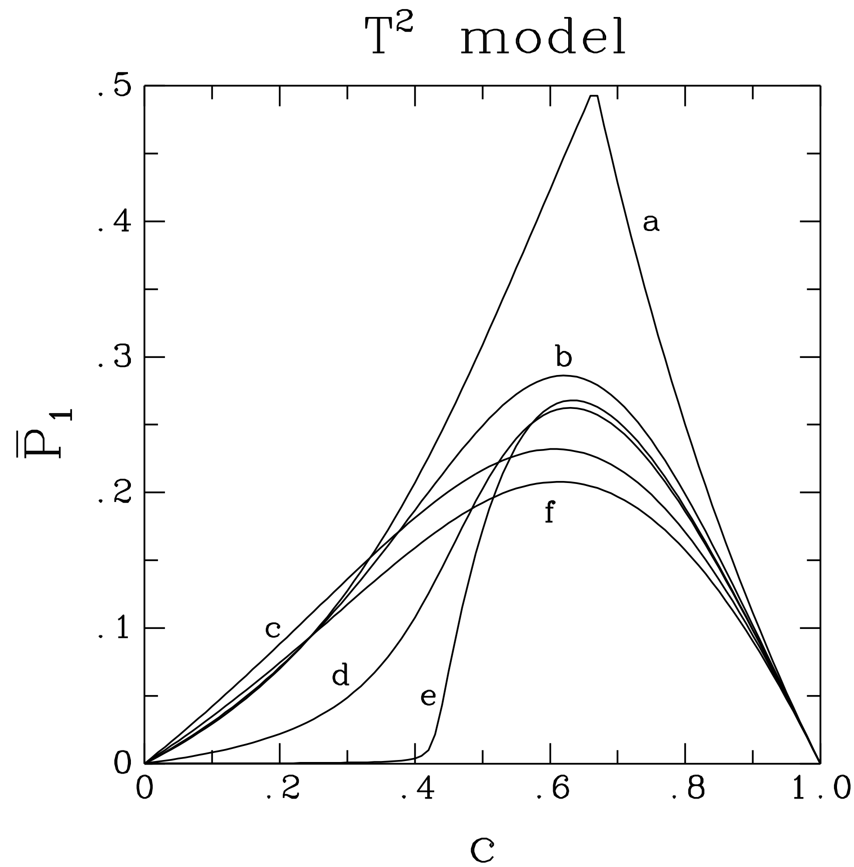

Figure 6: $\bar{P}_1$ for the T^2 model for a) $p = 0.5$, $p_s = 1$, b) $p = 0.5$, $p_s = 0.75$, c) $p = 0.75$, $p_s = 0.5$, d) $p = 0.1$, $p_s = 0.5$, e) $p = 0.001$, $p_s = 0.5$.